

CLOSED H -ORBITS IN G -VARIETIES

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ABSTRACT. Let G be a linear algebraic group, H a subgroup of G and X a G -variety. This paper explores the connection between G -orbits and H -orbits in X , concentrating in particular on the question of when we have the implications $G \cdot x$ closed in X implies $H \cdot x$ closed in X , and vice versa. Some of the general results found in this paper are then used in the special case where G and H are reductive and X is affine to extend two classical results of Luna from characteristic zero to arbitrary characteristic.

1. INTRODUCTION

Let k be an algebraically closed field of characteristic $p \geq 0$. Given a linear algebraic group G over k and a variety X on which G acts, one naturally wants to find the closed orbits for G in X . In some cases, there is a connection between the closed G -orbits in X and the closed H -orbits in X for certain subgroups H of G , see [19], [29], [2] and [6] for example. Motivated by results in these papers, in this note we shall explore the following basic problem: Suppose X is a G -variety, $x \in X$, and H is a subgroup of G . What conditions on G , X , x and H ensure that $G \cdot x$ is closed implies $H \cdot x$ is closed, and vice versa?

One of the author's original motivations for studying this question was the paper of Luna [19]. Suppose that $p = 0$, that G is reductive and that X is affine. Given $x \in X$ and a reductive subgroup A of the stabilizer of x , then [19, Cor. 1] says that $G \cdot x$ is closed if and only if $H \cdot x$ is closed, where $H = C_G(A)$ or $N_G(A)$. A partial extension of the forward implication of this result to the case where $p > 0$ was given in [2, Thm. 4.4(i)], but it was not obvious at the time if or how the other direction could be extended; this paper has grown in part out of attempts to find the right framework for fully extending Luna's result to positive characteristic. This aim is achieved in Corollary 4.3 below.

The work in this paper is part of the continuing process of trying to take results about algebraic groups and related structures from characteristic zero and prove analogues in positive characteristic, which has been a theme of this area of mathematics over so many years. A basic problem with this process is that results which are true when $p = 0$ may simply fail when $p > 0$; an illustration of this in the context of this paper is the difference between a group being reductive and linearly reductive (recall that an algebraic group is called *reductive* if it has trivial unipotent radical, and *linearly reductive* if all its rational representations are semisimple). When $p = 0$, a connected group is linearly reductive if and only if it is reductive, whereas if $p > 0$ a connected group is linearly reductive if and only if it is a torus, [24]. Even if a result remains true in positive characteristic, it may be much harder to prove, an example here being the problem of showing that the ring of invariants R^G is finitely generated, where R is a finitely-generated k -algebra and $G \subset \text{Aut}(R)$ is reductive. This was resolved in characteristic 0 in the 1950s, but not in positive characteristic until the 1970s (see the introduction to Haboush's paper [11]).

In some contexts in positive characteristic where the hypothesis of reductivity is too weak and linear reductivity is too strong, it has been found that a third notion, that of G -complete reducibility, provides a good balance. This notion was introduced by Serre [32] and Richardson¹ [30] and over the past 15 years or so has found many applications in the theory of algebraic groups, their subgroup structure and representation theory, geometric invariant theory, and the theory of buildings: for examples, see [2], [4], [8], [17], [20], [21], [22], [36]. The idea is that when $p = 0$ there is no difference between demanding that a (connected) subgroup H of a reductive group G is reductive or linearly reductive or G -completely reducible, but this may make a huge difference to the result in positive characteristic. The results presented in Section 4 of this paper are in this vein.

The paper is set out as follows. We begin in Section 2 with some notation and gathering of pertinent results from the literature, together with some fairly straightforward technical lemmas which are used

Date: November 6, 2012.

¹Richardson originally defined *strongly reductive* subgroups, but his notion was shown to be equivalent to Serre's in [4, Thm. 3.1].

in the sequel. Section 3 contains some results of a “general” nature, which means that, even if we are mainly interested in reductive groups and affine varieties, at least parts of the main results in this section hold for groups which are not necessarily reductive and/or varieties which are not necessarily affine. Also here are some illustrative examples. In Section 4, we first recap some material on G -complete reducibility before using our earlier results, together with some extra tools available for reductive groups acting on affine varieties, to get the extension of Luna’s results and some consequences in the setting of G -complete reducibility.

Acknowledgements: I’d like to thank Sebastian Herpel for the conversations we had which led to me first writing down some of the ideas in this paper, and Stephen Donkin for some very helpful nudges in the right direction during the latter stages of preparing the paper, in particular regarding the proof of Theorem 4.5.

2. NOTATION AND PRELIMINARIES

2.1. Notation. Our basic references for the theory of linear algebraic groups are the books [9] and [34]. Unless otherwise stated, we work over a fixed algebraically closed field k with no restriction on the characteristic. For a linear algebraic group G over k , we let G^0 denote the connected component of G containing the identity element 1 and $R_u(G) \subseteq G^0$ denote the unipotent radical of G . We say that G is *connected* if $G = G^0$ and G is *reductive* if G is connected and $R_u(G) = \{1\}$. When we discuss subgroups of G , we really mean *closed* subgroups; for two such subgroups H and K of G , we set $HK := \{hk \mid h \in H, k \in K\}$.

Given a linear algebraic group G , let $Y(G)$ denote the set of cocharacters of G , where a *cocharacter* is a homomorphism of algebraic groups $\lambda : k^* \rightarrow G$. Note that since the image of a cocharacter is connected, we have $Y(G) = Y(G^0)$. A linear algebraic group G acts on its set of cocharacters: for $g \in G$, $\lambda \in Y(G)$ and $t \in k^*$, we set $(g \cdot \lambda)(t) = g\lambda(t)g^{-1}$.

Given an algebraic variety X over k , we denote the coordinate ring of X by $k[X]$. We say that X is a G -variety if G acts morphically on X . The action of G on X gives a linear action of G on $k[X]$, defined by $(g \cdot f)(x) = f(g^{-1} \cdot x)$ for all $g \in G$, $f \in k[X]$ and $x \in X$. Given a G -variety X and $x \in X$, we denote the G -orbit through x by $G \cdot x$ and the stabilizer of x in G by G_x . If $x, y \in X$ are two points on the same G -orbit, then we sometimes say x and y are G -conjugate. We denote by X^G the set of G -fixed points in X , and by $k[X]^G$ the set of G -invariant functions in $k[X]$. For any cocharacter $\lambda \in Y(G)$ and $x \in X$ we can define a morphism $\psi = \psi_{x,\lambda} : k^* \rightarrow X$ by $\psi(t) = \lambda(t) \cdot x$ for each $t \in k^*$. We say that the *limit* $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ exists if ψ extends to a morphism $\bar{\psi} : k \rightarrow X$. If the limit exists, then the extension $\bar{\psi}$ is unique, and we set $\lim_{t \rightarrow 0} \lambda(t) \cdot x = \bar{\psi}(0)$. The use of limits to detect whether or not a G -orbit is closed is crucial in much of what follows, especially in the case of a reductive group acting on an affine variety. For now we record an easy result which is valid for arbitrary G and X .

Lemma 2.1. *Suppose G is a linear algebraic group and X is a G -variety. Let $x \in X$. If there exists $\lambda \in Y(G)$ such that $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ exists but lies outside $G \cdot x$, then $G \cdot x$ is not closed in X .*

Proof. Let $\bar{\psi} : k \rightarrow X$ be the morphism giving the limit. It is clear from the definition that $\bar{\psi}(t) \in G \cdot x$ for all $t \in k \setminus \{0\}$. If, however, $G \cdot x$ does not contain $\bar{\psi}(0) = \lim_{t \rightarrow 0} \lambda(t) \cdot x$, then $(\bar{\psi})^{-1}(G \cdot x) = k \setminus \{0\}$ is not closed in k , hence $G \cdot x$ is not closed in X . \square

Let G be a linear algebraic group over k and suppose H is a subgroup of G . Recall that the quotient G/H exists, has the structure of a homogeneous G -variety, and H is the stabilizer of the image of $1 \in G$ under the natural map $\pi_H : G \rightarrow G/H$. Recall also that the Zariski topology on G/H is the quotient topology; i.e., a subset $S \subseteq G/H$ is closed in G/H if and only if $\pi_H^{-1}(S)$ is closed in G . A subgroup P of G is called a *parabolic subgroup* if the quotient G/P is complete (if and only if it is projective). If G is reductive, then all parabolic subgroups of G have a *Levi decomposition* $P = R_u(P) \rtimes L$, where the reductive subgroup L is called a *Levi subgroup* of P . In this case, the unipotent radical $R_u(P)$ acts simply transitively on the set of Levi subgroups of P , and given a maximal torus T of P there exists a unique Levi subgroup of P containing T . For these standard results see [9], [10] or [34] for example.

Given an algebraic variety X and $x \in X$, we let $T_x(X)$ denote the tangent space to X at x . Recall that for a linear algebraic group G , $T_1(G)$ has the structure of a Lie algebra, which we also denote by $\text{Lie}(G)$. Given a morphism $\phi : X \rightarrow Y$ of algebraic varieties X and Y and a point $x \in X$, we let $d_x\phi : T_x(X) \rightarrow T_{\phi(x)}(Y)$ denote the differential of ϕ at x .

2.2. Group actions and categorical quotients. In the sequel we need some facts about the existence and properties of quotients of varieties by algebraic group actions. We only consider the case where H is a linear algebraic group, H^0 is reductive, and X is an affine H -variety. As noted above, H also acts on $k[X]$, and we can form the subring $k[X]^H \subseteq k[X]$ of H -invariant functions on X . It follows from [25] and [11] that $k[X]^H$ is finitely generated, and hence we can form an affine variety Y with coordinate ring $k[Y] = k[X]^H$. Moreover, the inclusion $k[X]^H \hookrightarrow k[X]$ gives rise to a morphism $\pi : X \rightarrow Y$ with the following properties [23, Thm. A.1.1], [26, Thm. 3.5], [1, §2]:

- (a) π is surjective;
- (b) π is constant on H -orbits in X ;
- (c) π separates disjoint closed H -invariant subsets of X ;
- (d) each fibre of π contains a unique closed H -orbit, and π determines a bijective map from the set of closed H -orbits in X to Y .

In the case that H is a subgroup of a reductive group G and H acts on G by $h \cdot g = gh^{-1}$, the variety Y is just the quotient G/H . Richardson has shown the following in this situation [28, Thm. A], see also [12].

Theorem 2.2. *Suppose H is a subgroup of G . Then G/H is an affine variety if and only if H^0 is reductive.*

2.3. Reductive groups, affine varieties and cocharacters. In this subsection let G be a reductive group and X an affine G -variety. We collect some results relating closed orbits in X to cocharacters of G . Recall the definition of the limit from above. We can recover the parabolic and Levi subgroups of G by considering the existence of suitable limits [34, Prop. 8.4.5]. In particular, given $\lambda \in Y(G)$, we have:

- (i) $P_\lambda := \{g \in G \mid \lim_{t \rightarrow 0} \lambda(t)g\lambda(t)^{-1} \text{ exists}\}$ is a parabolic subgroup of G ;
- (ii) $L_\lambda := C_G(\lambda) = \{g \in G \mid \lim_{t \rightarrow 0} \lambda(t)g\lambda(t)^{-1} = g\}$ is a Levi subgroup of P_λ ;
- (iii) $R_u(P_\lambda) = \{g \in G \mid \lim_{t \rightarrow 0} \lambda(t)g\lambda(t)^{-1} = 1\}$.

Moreover, given any parabolic subgroup P of a reductive group G and any Levi subgroup L of P , there exists $\lambda \in Y(G)$ such that $P = P_\lambda$ and $L = L_\lambda$. If H is a reductive subgroup of G and $\lambda \in Y(H)$, then λ gives rise in this way to parabolic and Levi subgroups of both G and H . We reserve the notation P_λ , L_λ , etc. for parabolic subgroups of G , and use the notation $P_\lambda(H)$, $L_\lambda(H)$, etc. to denote the corresponding subgroups of H . Note that for $\lambda \in Y(H)$, it is obvious from the definitions that $P_\lambda(H) = P_\lambda \cap H$, $L_\lambda(H) = L_\lambda \cap H$ and $R_u(P_\lambda(H)) = R_u(P_\lambda) \cap H$.

The classic Hilbert-Mumford Theorem [15, Thm. 1.4] says that via the process of taking limits, the cocharacters of G can be used to detect whether or not the G -orbit of a point in X is closed, extending the basic observation made in Lemma 2.1. Kempf strengthened the Hilbert-Mumford Theorem in [15] (see also [13], [23], [31]), by developing a theory of “optimal cocharacters” for non-closed G -orbits. We give an amalgam of some results from Kempf’s paper, see [15, Thm. 3.4, Cor. 3.5].

Theorem 2.3. *Let $x \in X$ be such that $G \cdot x$ is not closed. Then there exists a proper parabolic subgroup $P(x)$ of G and a class $\Omega(x) \subseteq Y(G)$ such that:*

- (i) *for all $\lambda \in \Omega(x)$, $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ exists and is not G -conjugate to x ;*
- (ii) *for all $\lambda \in \Omega(x)$, $P_\lambda = P(x)$;*
- (iii) *$R_u(P(x))$ acts simply transitively on $\Omega(x)$;*
- (iv) *$G_x \subseteq P(x)$.*

In the paper [8], many general results were proved about the orbits of a reductive group G in an affine G -variety, leading to the following very useful strengthening of the Hilbert-Mumford Theorem, which we use a few times in this paper.

Theorem 2.4. *Let $x \in X$. Then $G \cdot x$ is closed if and only if $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ is $R_u(P_\lambda)$ -conjugate to x for all $\lambda \in Y(G)$ such that the limit exists.*

Proof. The Hilbert-Mumford Theorem (or Kempf’s strengthening of it given in Theorem 2.3 above) implies that $G \cdot x$ is closed if and only if whenever the limit $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ exists it is G -conjugate to x . But [8, Thm. 3.3] says that this limit is G -conjugate to x if and only if it is $R_u(P_\lambda)$ -conjugate to x , which gives the result. \square

This result shows that if we are interested in whether or not an orbit is closed, we are interested in taking limits along cocharacters, and conjugacy under unipotent radicals of parabolics. The final preparatory lemmas of this subsection provide some useful tools in this set-up. The first is [8, Lem. 2.12].

Lemma 2.5. *Suppose $x' = \lim_{t \rightarrow 0} \lambda(t) \cdot x$ exists. Then $x' = u \cdot x$ for $u \in R_u(P_\lambda)$ if and only if $u^{-1} \cdot \lambda$ fixes x .*

Lemma 2.6. *Suppose $x' = \lim_{t \rightarrow 0} \lambda(t) \cdot x$ exists, let $u \in R_u(P_\lambda)$, and set $\mu = u \cdot \lambda$. Then $\lim_{t \rightarrow 0} \mu(t) \cdot x$ exists and is equal to $u \cdot x'$.*

Proof. We note that

$$\mu(t) \cdot x = (u\lambda(t)u^{-1}) \cdot x = (u\lambda(t)u^{-1}\lambda(t)^{-1}\lambda(t)) \cdot x = u(\lambda(t)u^{-1}\lambda(t)^{-1}) \cdot (\lambda(t) \cdot x),$$

for all $t \in k^*$. Now $\lim_{t \rightarrow 0} \lambda(t)u^{-1}\lambda(t)^{-1} = 1$ since $u \in R_u(P_\lambda)$, and $\lim_{t \rightarrow 0} \lambda(t) \cdot x = x'$. Since the displayed equality holds for all $t \in k^*$, and these two limits exist, we can conclude that $\lim_{t \rightarrow 0} \mu(t) \cdot x$ exists and equals $u \cdot x'$, as required. \square

Our final result is a useful structural result about reductive subgroups of reductive groups and their parabolic subgroups.

Lemma 2.7. *Suppose $H \subseteq G$ are reductive groups, and suppose $\lambda, \mu \in Y(H)$ are $R_u(P_\lambda)$ -conjugate. Then λ and μ are $R_u(P_\lambda(H))$ conjugate. Moreover, if $u \in R_u(P_\lambda)$ is such that $\mu = u \cdot \lambda$, then in fact $u \in R_u(P_\lambda(H))$.*

Proof. Suppose $\mu = u \cdot \lambda$ for some $u \in R_u(P_\lambda)$. Then $P_\lambda = P_\mu$, and L_λ and $L_\mu = uL_\lambda u^{-1}$ are Levi subgroups of P_λ . We also have that $L_\lambda(H)$ and $L_\mu(H)$ are Levi subgroups of $P_\lambda(H)$, so they are conjugate under $R_u(P_\lambda(H)) = R_u(P_\lambda) \cap H$. Let $v \in R_u(P_\lambda(H))$ be such that $vL_\lambda(H)v^{-1} = L_\mu(H)$. Then we claim that $vL_\lambda v^{-1} = L_\mu$. To see this, note that $\mu \in Y(vL_\lambda v^{-1})$, so there exists some maximal torus T of $vL_\lambda v^{-1}$ with $\mu \in Y(T)$. But then $T \subseteq C_G(\mu) = L_\mu$, so that L_μ and $vL_\lambda v^{-1}$ are Levi subgroups of P_λ containing a common maximal torus. Thus $L_\mu = vL_\lambda v^{-1}$, as required. But now we have $L_\mu = vL_\lambda v^{-1} = uL_\lambda u^{-1}$. Since $R_u(P_\lambda)$ acts simply transitively on the set of Levi subgroups of P_λ , we have $u = v$, which gives both statements of the result. \square

3. GENERAL RESULTS

In this section we prove our main results concerning the implication “ $G \cdot x$ closed implies $H \cdot x$ closed”. For the whole section, let G be a connected linear algebraic group and X a G -variety. Before proving the first main result we need a technical lemma which collects together various properties of orbits and quotients and the associated morphisms. Parts of the result are probably well known – see the proofs of [29, Lem. 4.2, Lem. 10.1.3] or the discussion in [14, Sec. 2.1], for example – but we include a proof for completeness.

Lemma 3.1. *Suppose $x \in X$, and let $\psi_x : G/G_x \rightarrow G \cdot x$ be the natural map. Then:*

- (i) ψ_x is a homeomorphism;
- (ii) $G \cdot x$ is affine if and only if G/G_x is affine;
- (iii) ψ_x is an isomorphism of varieties if and only if the orbit map $\phi_x : G \rightarrow G \cdot x$ is separable.

Proof. (i). It is standard group theory that the natural map ψ_x defined by $\psi_x(gG_x) = g \cdot x$ from the quotient to the orbit is a bijective morphism. Since G is irreducible, the homogeneous G -varieties G/G_x and $G \cdot x$ are irreducible, and hence normal by [34, Cor. 5.3.4]. Now the fibres of ψ_x are singletons, so ψ_x is an open map by [9, Cor. 18.4]. But an open map of topological spaces which is a bijection is a homeomorphism.

(ii). By [9, Thm. 18.2], ψ_x is a normalization of $G \cdot x$. Since the normalization of an affine variety is affine, it follows that if $G \cdot x$ is affine then so is the quotient G/G_x . On the other hand, if G/G_x is affine, then so is $G \cdot x$ by [9, Prop. 18.3].

(iii). This follows from the fact that the orbit map $\phi_x : G \rightarrow G \cdot x$ is separable if and only if its differential $d_1\phi_x : \text{Lie}(G) \rightarrow T_x(G \cdot x)$ is surjective, and this is equivalent to the differential $d_{\pi_H(1)}\psi_x : T_{\pi_H(1)}(G/G_x) \rightarrow T_x(G \cdot x)$ being bijective. By [34, Thm. 5.3.2], this is equivalent to ψ_x being an isomorphism. \square

Remark 3.2. All the subtleties here are only really important in positive characteristic since in characteristic 0 the orbit map is always separable, so the morphism ψ_x is always an isomorphism. The result shows that even in bad cases where the orbit map is not separable we can reasonably compare the quotient G/G_x with the orbit $G \cdot x$, as you might hope.

Theorem 3.3. *Let H be a subgroup of G and suppose $x \in X$. Set $K = G_x$ and let H act on X by restriction of the G -action. Then:*

- (i) $H \cdot x$ is closed in $G \cdot x$ if and only if $HK = \{hk \mid h \in H, k \in K\}$ is a closed subset of G ;
- (ii) If $G \cdot x$ is closed in X then $H \cdot x$ is closed in X if and only if HK is closed in G .

Proof. (i). Since the map $\psi_x : G/K \rightarrow G \cdot x$ from Lemma 3.1 is a homeomorphism, $H \cdot x$ is closed in $G \cdot x$ if and only if the corresponding subset $H \cdot \pi_K(1)$ is closed in G/K . Since the topology on the quotient is the quotient topology, this is the case if and only if the preimage of this orbit is closed in G . But the preimage is precisely the subset HK .

(ii). If $G \cdot x$ is itself closed, then the statement in part (ii) follows from part (i) because the Zariski topology on $G \cdot x$ is the subspace topology and a closed subset of a closed subset is closed. \square

Remarks 3.4. (i). One can give a direct proof of Theorem 3.3, using the fact that the orbit map $G \rightarrow G \cdot x$ is open, but going via the quotient using Lemma 3.1 perhaps makes it more transparent what is going on.

(ii). Note that the forward implication in Theorem 3.3(ii) above holds without the closedness assumption on $G \cdot x$: If $H \cdot x$ is closed in all of X then $H \cdot x$ must be closed in $G \cdot x$, so HK is closed in G by Theorem 3.3(i).

Corollary 3.5. *With notation as in Theorem 3.3, suppose H normalizes K or K normalizes H . Then $H \cdot x$ is closed in $G \cdot x$. In particular, if $G \cdot x$ is closed in X then so is $H \cdot x$.*

Proof. This is clear from the theorem, since if K normalizes H or H normalizes K then the product HK is actually closed subgroup of G . \square

Remark 3.6. Note that Corollary 3.5 applies in the special case that $K \subseteq H$. Actually, it is enough to know that $K^0 \subseteq H$, since then HK^0 is closed and HK is a finite union of closed sets HK^0k where $k \in K$ runs over representatives of the right cosets of K^0 in K .

Our next result involves the following set-up: Suppose Y is another G -variety. Then $G \times G$ acts on the product $X \times Y$ via $(g_1, g_2) \cdot (x, y) = (g_1 \cdot x, g_2 \cdot y)$, and identifying G with its diagonal embedding $\Delta(G)$ in $G \times G$, we can also get the diagonal action of G on $X \times Y$: $g \cdot (x, y) = (g \cdot x, g \cdot y)$.

Theorem 3.7. *With the notation just introduced, let $x \in X$, $y \in Y$ and set $K = G_x$, $H = G_y$. Then:*

- (i) $H \cdot x$ is closed in $G \cdot x$ if and only if $K \cdot y$ is closed in $G \cdot y$ if and only if $G \cdot (x, y)$ is closed in $(G \cdot x) \times (G \cdot y)$;
- (ii) If $G \cdot x$ is closed in X and $G \cdot y$ is closed in Y , then $H \cdot x$ is closed in X if and only if $K \cdot y$ is closed in Y if and only if $G \cdot (x, y)$ is closed in $X \times Y$.

Proof. (i). The first equivalence follows from Theorem 3.3 since $KH = (HK)^{-1}$ is closed in G if and only if HK is closed in G (note that this argument is based on the one in the proof of [29, Lem. 10.1.4]).

For the second equivalence, consider the map $\pi : G \times G \rightarrow G$ given by $\pi(g_1, g_2) = g_1^{-1}g_2$. Then π is a surjective morphism. Moreover, given $g \in G$, the associated fibre

$$\pi^{-1}(g) = \{(g_1, g_2) \mid g_1^{-1}g_2 = g\} = \{(g_1, g_1g) \mid g_1 \in G\} = \Delta(G)(1, g)$$

is a coset of $\Delta(G)$ in $G \times G$. In particular, all fibres have the same dimension, so π is an open map by [9, Cor. 18.4]. Now, since the $(G \times G)$ -orbit of (x, y) is $(G \cdot x) \times (G \cdot y)$ and the stabilizer of (x, y) in $G \times G$ is $K \times H$, we have that $\Delta(G) \cdot (x, y)$ is closed in $(G \cdot x) \times (G \cdot y)$ if and only if $\Delta(G)(K \times H)$ is closed in $G \times G$, by Theorem 3.3(i). Now it is easy to see that $\Delta(G)(K \times H) = \pi^{-1}(KH)$, and since π is a surjective open map, we have $\Delta(G)(K \times H)$ is closed in $G \times G$ if and only if KH is closed in G , which happens if and only if $K \cdot y$ is closed in $G \cdot y$, by Theorem 3.3(i) again.

(ii). This chain of equivalences follows quickly from part (i), together with our previous observation that a closed subset of a closed subset is closed and the fact that if $G \cdot x$ is closed in X and $G \cdot y$ is closed in Y , then $(G \cdot x) \times (G \cdot y)$ is closed in $X \times Y$. \square

Remark 3.8. As we indicated at the start of this section, the results above give criteria for “ $G \cdot x$ closed implies $H \cdot x$ closed”. Here is an easy example in the context of Theorem 3.7 which shows that we can’t hope for a general converse to this. Let $G = \mathrm{GL}_2$ and let $X = Y = G^2$, with G acting via simultaneous conjugation on each factor. Let $1 \neq a \in k^*$ and set

$$x = \left(\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right), \quad y = \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right).$$

Then a quick calculation shows that $G_x = G_y = Z(G)$, so that it is trivially true that $G_x \cdot y$ and $G_y \cdot x$ are closed because the orbits are singletons. It is also true that $G \cdot y$ and $G \cdot (x, y)$ are closed in Y and

$X \times Y$ respectively (for example because the subgroups of G generated by the entries of the tuples y and (x, y) are G -cr, see subsection 4.1 below). However, $G \cdot x$ is not closed. To see this, consider, for example, the cocharacter $\lambda \in Y(G)$ defined by

$$\lambda(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \quad \text{for } t \in k^*.$$

Then an easy calculation shows that

$$\lim_{t \rightarrow 0} \lambda(t) \cdot x = \lim_{t \rightarrow 0} \left(\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} 1 & t^2 \\ 0 & 1 \end{pmatrix} \right) = \left(\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right).$$

It is clear that this limit is not conjugate to x , so the G -orbit of x is not closed. Note that in this example all the groups are reductive and the varieties are affine.

A similar example can be constructed for any reductive group G : for example, in the language of subsection 4.1 below, let $x \in G^n$ be a generic tuple for a Borel subgroup of G , let $y \in G^n$ be a generic tuple for G itself. Then, as above, all stabilizers are central and the G -orbits of y and (x, y) are closed, but the G -orbit of x is not closed.

We finish the section with some stronger results which hold in the special case that G is a reductive group. The first thing this assumption allows us to do is apply Richardson's Theorem 2.2 to the quotient G/G_x .

Lemma 3.9. *Suppose G is reductive, X is a G -variety and $x \in X$. Then G_x^0 is reductive if and only if $G \cdot x$ is affine.*

Proof. By Lemma 3.1(ii), $G \cdot x$ is affine if and only if G/G_x is affine, and this happens if and only if G_x^0 is reductive, by Theorem 2.2. \square

Remark 3.10. Note that this result is an extension of Richardson's result [29, Lem. 10.1.3], which essentially gives the reverse implication under the assumption that X is affine and $G \cdot x$ is closed in X . It is claimed in [23, Appendix 1D] that Haboush [12] and Richardson [28] give proofs of Lemma 3.9, but these references only seem to contain the result of Theorem 2.2. It is not obvious that this result is enough without the additional input from Lemma 3.1(ii).

Theorem 3.11. *Suppose G is reductive, X is a G -variety and $x \in X$ is such that $G \cdot x$ is affine. If H is a reductive subgroup of G containing a maximal torus of G_x , then $H \cdot x$ is closed in $G \cdot x$. In particular, if X is affine and $G \cdot x$ is closed in X , then $H \cdot x$ is closed in X .*

Proof. Suppose, for contradiction, that $H \cdot x$ is not closed in $G \cdot x$. Then, by Kempf's Theorem 2.3, since $G \cdot x$ is an affine H -variety, there exists an optimal cocharacter $\lambda \in Y(H)$ such that $x' = \lim_{t \rightarrow 0} \lambda(t) \cdot x$ exists in $G \cdot x$ but is not H -conjugate to x , and $H_x \subseteq P_\lambda(H)$. Let S be a maximal torus of G_x contained in H . Then $S \subseteq H_x$, so S is contained in P_λ . Since S is a maximal torus of G_x , S is also a maximal torus of $P_\lambda \cap G_x$.

Since $x' \in G \cdot x$, it follows from Theorem 2.4 and Lemma 2.5 that there exists $u \in R_u(P_\lambda)$ such that $x' = u \cdot x$ and $\mu = u^{-1} \cdot \lambda$ fixes x , i.e. $\mu \in Y(P_\lambda \cap G_x)$. Since S is a maximal torus of $P_\lambda \cap G_x$, there exists $g \in P_\lambda \cap G_x$ such that $g \cdot \mu \in Y(S)$. Since $P_\lambda = P_\mu = R_u(P_\lambda) \rtimes L_\mu$, we can write $g = u_1 l$, where $u_1 \in R_u(P_\lambda)$ and $l \in L_\mu$. But everything in L_μ commutes with μ , so $g \cdot \mu = (u_1 l) \cdot \mu = u_1 \cdot \mu$. Now we have λ and $(u_1 u^{-1}) \cdot \lambda \in Y(H)$, so by Lemma 2.7 $u_1 u^{-1} \in H$, and hence $u u_1^{-1} \in H$. Since $u_1 \in G_x$, we also have $u u_1^{-1} \cdot x = u \cdot x = x'$. This contradicts the fact that x' is not H -conjugate to x , hence we must conclude that $H \cdot x$ is closed in $G \cdot x$, as required.

For the final part, in the special case that X is affine the assumption that $G \cdot x$ is closed in X implies that $G \cdot x$ is affine. Now we get straight away that $H \cdot x$ is also closed in X . \square

Remark 3.12. Lemma 3.9 shows that the assumption that $G \cdot x$ is affine means that the connected stabilizer G_x^0 is reductive. We have already noted in Remark 3.6 that the first part of the theorem holds without the assumption that G or H are reductive if the subgroup H contains all of G_x^0 . The theorem above shows if H and G are reductive, then it is enough to know that $H_x = H \cap G_x$ has maximal rank in G_x ; note that in this case, since $H \cdot x$ is a closed subset of the affine variety $G \cdot x$, $H \cdot x$ is also affine, so H_x^0 is reductive, by Lemma 3.9.

Our next results give some structural results about G and its subgroups which can quickly be proven using the framework we have now set up.

Corollary 3.13. *Suppose U is a closed unipotent subgroup of G and let K be any closed subgroup of G with K^0 reductive. Then UK is a closed subset of G .*

Proof. The quotient $X = G/K$ is affine and K is the stabilizer in G of the point $x = \pi_K(1) \in X$. Since U is unipotent, and all orbits for unipotent groups on affine varieties are closed [9, Prop. 4.10], we have $U \cdot x$ is closed, so UK is closed in G by Theorem 3.3. \square

Corollary 3.14. *Suppose H and K are reductive subgroups of the reductive group G . If $H \cap K$ contains a maximal torus of H or K , then HK is closed in G and $(H \cap K)^0$ is a reductive group.*

Proof. Without loss, suppose $H \cap K$ contains a maximal torus of K . The quotient variety $X = G/K$ is affine, so applying Theorem 3.11 and Theorem 3.3 to X and the point $x = \pi_K(1)$ gives that HK is closed in G . The second assertion follows using the argument at the end of Remark 3.12: note that G/K is affine, and hence the closed orbit $H \cdot x$ is affine; now apply Lemma 3.9 to the stabilizer $H_x = H \cap K$. \square

4. APPLICATIONS AND CONVERSES

In this section we apply the results of the previous section to interesting special cases, and also explore to what extent we can get results in the other direction “ $H \cdot x$ closed implies $G \cdot x$ closed”. We also use some of the special cases to illustrate the failure of our results under weaker hypotheses. Many of our results use the framework of G -complete reducibility introduced by J-P. Serre [32], which has been shown to have geometric implications in [4] and subsequent papers. We therefore begin the section with a short recap of some of the key ideas concerning complete reducibility.

4.1. G -complete reducibility. Let G be a reductive linear algebraic group, and K a subgroup of G . Following Serre (see, for example, [32]), we call K *G -completely reducible* (G -cr for short) if whenever $K \subseteq P$ for a parabolic subgroup P of G , there exists a Levi subgroup L of P such that $K \subseteq L$. It turns out that this notion serves to classify closed orbits in the affine G -variety G^n , where G acts on G^n by simultaneous conjugation; see [4] for details. In [2] it was shown that the notion of complete reducibility is useful when one considers other G -actions and, as explained in the introduction, one of the purposes of this section of the current paper is to continue that theme and expand upon it.

It is convenient in what follows to have a slightly more general notion of complete reducibility valid for groups which are not connected, but whose unipotent radical is trivial; this allows us both to prove more general results and to avoid the notational problems which come with having to take identity components all the time. We call a linear algebraic group H such that $R_u(H) = 1$ a *non-connected reductive group*, and we recall from [4, Sec. 6] that one can define a notion of H -complete reducibility for subgroups of H which reduces to the usual notion in case $H = H^0$. The essential trick is to restrict attention to those parabolic subgroups of H which are of the form P_λ , where $\lambda \in Y(H)$ (these are called *R -parabolic subgroups* in [4]). This no longer covers *all* the parabolic subgroups of H in general. However, once this notion is correctly defined, [4, Sec. 6] shows that all the results which are valid for connected groups go through in this slightly more general setting. We omit the details.

The geometric approach to complete reducibility outlined in [4] and subsequent papers rests on the following construction, which is first given in its current form in [8]. Given a subgroup K of a reductive group G and a positive integer n , we call a tuple of elements $\mathbf{k} \in K^n$ a *generic tuple* for K if there exists an embedding of G in some GL_m such that \mathbf{k} generates the associative subalgebra of $m \times m$ matrices spanned by K [8, Defn. 5.4]. A generic tuple for K always exists for sufficiently large n . Suppose $\mathbf{k} \in K^n$ is a generic tuple for K , then in [8, Thm. 5.8(iii)] it is shown that K is G -completely reducible if and only if the G -orbit of \mathbf{k} in G^n is closed, where G acts on G^n by simultaneous conjugation.

4.2. Two results from [19]. Our first result provides a partial converse to some of the results in Section 3, as it addresses the problem of when “ $H \cdot x$ closed implies $G \cdot x$ closed”. The result is a generalization of [2, Thm. 4.4]; see also [6, Thm. 5.4].

Theorem 4.1. *Suppose that G is a reductive group and X is an affine G -variety. Let $x \in X$, let $A \subseteq G_x$ be G -completely reducible and let H be any subgroup of G containing $C_G(A)^0$. Then if $H \cdot x$ is closed in X , $G \cdot x$ is closed in X .*

Proof. We prove the contrapositive, so suppose $G \cdot x$ is not closed. Let $P(x)$ and $\Omega(x)$ be the parabolic subgroup and class of cocharacters given by Kempf’s Theorem 2.3. Since $A \subseteq G_x \subseteq P(x)$ is G -cr, there exists a Levi subgroup L of $P(x)$ containing A . Since $R_u(P(x))$ acts simply transitively on $\Omega(x)$ and on the set of Levi subgroups of $P(x)$, there exists $\lambda \in \Omega(x)$ with $L = L_\lambda$. But then $A \subseteq L_\lambda$ means that $\lambda \in Y(C_G(A)^0) \subseteq Y(H)$; in particular, $\lambda(t) \cdot x \in H \cdot x$ for all $t \in k^*$. Now $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ exists in

X and is not G -conjugate to x , so it is not H -conjugate to x , so $H \cdot x$ is not closed by Lemma 2.1, as required. \square

Remarks 4.2. (i). Note that we need the group G to be reductive and the variety X to be affine in order to apply Kempf's Theorem 2.3. We do not require the subgroup H in Theorem 4.1 to be reductive, although the subgroup $C_G(A)^0$ is reductive [4, Prop. 3.12], and this is the subgroup which is really controlling things in the proof.

(ii). In characteristic 0, the subgroup A of G is G -cr if and only if A^0 is reductive (cf. [4, §2.2 and Lem. 2.6]). In this case, therefore, we are just requiring that A is reductive. Therefore, when $\text{char } k = 0$ and $H = C_G(A)$ or $N_G(A)$, we retrieve one direction of Luna's result [19, §3, Cor. 1]. See Corollary 4.3 below for a generalization of both directions of Luna's result to arbitrary characteristic.

(iii). The reverse implication of Theorem 4.1 is not true in general, as the following example shows. This example, which is a modification of [2, Ex. 4.6], also shows that the result of Luna cited in (ii) fails in positive characteristic without extra hypotheses. Let the characteristic of k be 2 and let $G = \text{Sp}_8$ be the symplectic group in dimension 8. In [5, Ex. 5.3], there is an example of commuting connected subgroups A and B of G such that A and B are G -cr, but AB is not G -cr. By [5, Prop. 3.9], the fact that AB is not G -cr means that B is not $C_G(A)$ -cr. Let $H = C_G(A)$. For $n \in \mathbb{N}$ sufficiently large there exists a generic tuple $\mathbf{b} \in G^n$ for B (see subsection 4.1), and B is G -cr (resp. H -cr) if and only if the G -orbit of \mathbf{b} (resp. the H -orbit of \mathbf{b}) is closed in G^n (resp. H^n), where G and H act by simultaneous conjugation. Therefore, we can conclude in our example that $G \cdot \mathbf{b}$ is closed in G^n , but $H \cdot \mathbf{b}$ is not closed in H^n , and hence is not closed in G^n . Now if we set $X = G^n$, $x = \mathbf{b}$, then G , X , x , A and H satisfy the hypotheses of Theorem 4.1.

One can layer on extra hypotheses to get the sort of equivalence " $G \cdot x$ closed if and only if $H \cdot x$ " closed in many ways. The following example of such a result provides a condition which in particular rectifies the problem thrown up by the example in Remark 4.2(iii) above.

Corollary 4.3. *Suppose X is an affine G -variety and let $x \in X$. Suppose A is a subgroup of G_x and suppose $H = C_G(A)$. Then:*

- (i) *If A is G -cr, then $H \cdot x$ is closed in X implies $G \cdot x$ is closed in X ;*
- (ii) *If $G \cdot x$ is closed in X , then $H \cdot x$ is closed in X if and only if A is G_x -cr.*

Proof. (i). This is a special case of Theorem 4.1.

(ii). Set $K = G_x$. Since we are assuming $G \cdot x$ is closed, we know that K^0 is reductive, by Lemma 3.1, so it makes sense to ask whether or not A is K -cr (in the more general setting of non-connected reductive groups described in subsection 4.1). Now let $\mathbf{a} \in G^n$ for some n be a generic tuple for the subgroup A . Then $H = C_G(A) = G_{\mathbf{a}}$. Now, by Theorem 3.7, since $G \cdot x$ is closed in X , we have that $H \cdot x$ is closed in X if and only if $K \cdot \mathbf{a}$ is closed in G^n . Since K is a closed subgroup of G , $K \cdot \mathbf{a}$ is closed in G^n if and only if it is closed in K^n if and only if A is K -cr, as required. \square

Remarks 4.4. (i). In characteristic 0 the complete reducibility conditions on A in parts (i) and (ii) above are equivalent to requiring that A^0 is reductive, so we see that Corollary 4.3 is a generalization of Luna's result [19, §3, Cor. 1] to arbitrary characteristic.

(ii). The proof of Corollary 4.3(ii) goes through with a weaker assumption on A : one doesn't need to assume that A is contained in G_x if one is willing to use the notion of *relative complete reducibility* introduced in [7]. The new statement would read: If $G \cdot x$ is closed in X , then $H \cdot x$ is closed in X if and only if A is relatively G -cr with respect to G_x . We leave the details to the reader; the crucial point is that, by Theorem 3.7, we only care whether or not the G_x -orbit of the generic tuple \mathbf{a} for A is closed, and this is precisely what the notion of relative complete reducibility captures.

It is not clear, however, whether one can push the other direction of Corollary 4.3 further by using relative complete reducibility. Part (i) of the corollary relies on Theorem 4.1, which in turn relies on Kempf's Theorem 2.3 to provide a parabolic subgroup of G containing G_x . If A is no longer a subgroup of G_x , then there are obstructions to using this line of proof.

We finish this section by generalizing another result from Luna's paper to arbitrary characteristic, [19, Cor. 3]. The proof is similar in spirit to Luna's original proof – see Remark 4.6(i) below – but it is rather more involved because at various points we need to be careful to make sure the arguments go through.

Theorem 4.5. *Suppose G is a reductive group and H is a subgroup of G . Consider the following conditions on H :*

- (i) for every affine G -variety X , all G -orbits in X which meet X^H are closed;
- (ii) $N_G(H)/H$ is a finite group.

If H^0 is reductive, then (i) \implies (ii). If H is G -cr, then (i) and (ii) are equivalent.

Proof. Suppose (i) holds and H^0 is reductive. First note that, by [20, Lem. 6.8], H^0 being reductive implies that $N_G(H)^0 = H^0 C_G(H)^0$. Let $x \in C_G(H)$ be any element of the centralizer of H and let G act on itself by conjugation. We have $x \in C_G(H) = G^H$, so the G -orbit of x (i.e., the conjugacy class of x) must be closed in G . Hence x is a semisimple element of G [35, Cor. 3.6]. Since $C_G(H)$ consists entirely of semisimple elements, the identity component $C_G(H)^0$ must be a torus [9, Cor. 11.5(1)]. Hence $N_G(H^0) = H^0 C_G(H)^0$ is a reductive group and $(N_G(H)/H)^0$ is a torus. Suppose that $N_G(H)/H$ is infinite. Then there exists a one-dimensional subtorus S of $C_G(H)^0$ not contained in H . We construct a variety which contains H -fixed points whose G -orbits are not closed. To ease notation, let $Z = HS$ and note that $Z^0 = H^0 S$ is reductive.

Consider the action of Z on G given by $z * g = gz^{-1}$ for all $z \in Z, g \in G$. This gives a corresponding action of Z on $k[G]$, and we let S and H act by restriction of this action. The set $A := k[G]^H$ of H -invariants is finitely-generated since H^0 is reductive, and S -stable since if $f \in A, s \in S$ and $h \in H$, then $hs = sh$ so that

$$h * (s * f) = (hs) * f = (sh) * f = s * (h * f) = s * f \in A.$$

Note that $A \simeq k[G/H]$. Let $I = \{f \in A \mid f(h) = 0 \text{ for all } h \in H\}$ and $J = \{f \in A \mid f(z) = 0 \text{ for all } z \in Z\}$. Note that $J \subset I$, but $J \neq I$ since $Z \neq H$. As ideals of A , I and J are also finitely-generated. Choose a finite generating set for I . Then by [9, Prop. 1.9], there exists a finite-dimensional subspace E of A which is S -stable and contains these generators. Since E is an S -module and S is a torus, E decomposes $E = E^S \oplus M$, where M is the sum of all the non-trivial S -submodules of E . We claim that $M \not\subseteq I$. To see this, suppose the converse, i.e. that $m(h) = 0$ for all $m \in M$ and $h \in H$. Let $f \in I$ be any of the generators we have chosen, then $f \in E$ and we can write $f = f_1 + f_2$ with $f_1 \in E^S$ and $f_2 \in M$. Because M is S -stable and we are assuming $M \subseteq I$, we have $f_2(hs) = (s * f_2)(h) = 0$ for all $s \in S, h \in H$. Now $0 = f(h) = f_1(h) + f_2(h) = f_1(h)$, so since $f_1 \in E^S$, $f_1(hs) = (s * f_1)(h) = f_1(h) = 0$ for all $s \in S, h \in H$. But then we have $f(hs) = f_1(hs) + f_2(hs) = 0$ for all $s \in S, h \in H$ and hence $f \in J$. But this puts all generators for I inside J , and we have $I = J$, which is a contradiction. Hence our assumption that $M \subseteq I$ was false, which allows us to pick a function $m \in M$ with $m(h) \neq 0$ for at least one $h \in H$. Now M decomposes as a sum of non-trivial one-dimensional weight spaces for S , so we can project m onto these weight spaces and at least one component will still be non-zero at at least one $h \in H$. Moreover, since all functions in A are H -invariant, this function will be nonzero at all $h \in H$. I.e., we can find a function $m_\chi \in M$ such that:

- (a) $m_\chi(h) \neq 0$ for all $h \in H$;
- (b) $(hs) * m_\chi = \chi(s)m$ for all $h \in H, s \in S$, where $\chi : S \rightarrow k^*$ is a non-trivial weight of S .

Note that (b) follows since $hs = sh$ for all $h \in H$ and $s \in S$, and we were already working in the algebra A of H -fixed functions.

Let V be a one-dimensional vector space over k and let S act linearly on V with weight χ , where χ is the same weight as in the previous paragraph, so $s * v = \chi(s)v$ for each $v \in V$. Extend to an action of $Z = HS$ by letting H act trivially. Then Z has two orbits $\{0\}$ and $V \setminus \{0\}$ in V . Identify the coordinate ring $k[V]$ with the polynomial ring $k[T]$ in one indeterminate T , and note that, by construction, we have

$$((hs) * T)(v) = T((hs)^{-1} * v) = T(\chi(s)^{-1}v) = (\chi(s)^{-1}T)(v)$$

for all $h \in H, s \in S, v \in V$. Hence $(hs) * T = \chi(s)^{-1}T$ for all $h \in H, s \in S$. Let $X = G \times V$, and let Z act on X via

$$z * (g, v) = (z * g, z * v),$$

for $z \in Z, g \in G$ and $v \in V$. Since Z^0 is reductive, we can form the quotient Y of X by the action of Z as outlined in subsection 2.2. Recall that Y is affine, $k[Y] = k[X]^Z$, and the inclusion $k[X]^Z \subseteq k[X]$ induces a surjective morphism $\pi : X \rightarrow Y$ which is constant on Z -orbits in X and separates disjoint closed Z -invariant subsets of X .

G acts naturally on itself by left multiplication, and hence on X via

$$g \cdot (g', a) = (gg', a).$$

Since the G -action on X commutes with the Z -action, we can pass down to the quotient and make Y a G -variety. Concretely, we can write $g \cdot \pi(x) := \pi(g \cdot x)$ for $g \in G, x \in X$. Let $0 \neq v \in V$. We claim that the point $y = \pi(1, v) \in Y$ is H -fixed but has a non-closed G -orbit. We show this in several steps:

Step 1. y is H -fixed, since $h \cdot y = \pi(h \cdot (1, v)) = \pi(h^{-1} * (1, v)) = \pi(1, v) = y$ for all $h \in H$.

Step 2. The closure of the G -orbit of y contains $y_0 := \pi(1, 0)$, since for any $s \in S$, we have

$$s \cdot y = \pi(s, v) = \pi(s^{-1} * (1, s * v)) = \pi(1, s * v)$$

and the closure of the S -orbit of v in V contains 0.

Step 3. We have $y_0 \neq y$. To see this, recall the function $m_\chi \in A$ with properties (a) and (b) defined earlier in the proof. Then $m_\chi \otimes T \in k[Y] = k[G] \otimes k[T]$, and

$$(hs) * (m_\chi \otimes T) = ((hs) * m_\chi) \otimes ((hs) * T) = \chi(s)\chi(s)^{-1}m_\chi \otimes T = m_\chi \otimes T$$

for all $h \in H$ and $s \in S$. Hence $m_\chi \otimes T \in k[X]^Z = k[Y]$. But $m_\chi(h) \neq 0$ for all $h \in H$, so in particular $m_\chi(1) \neq 0$. So we have $(m_\chi \otimes T)(y_0) = m_\chi(1)T(0) = 0$ and $(m_\chi \otimes T)(y) = m_\chi(1)T(v) \neq 0$. Hence $y_0 \neq y$.

Step 4. y and y_0 are not G -conjugate. First note that if $g \in Z$, then $g \cdot y_0 = \pi(g, 0) = \pi(g^{-1} * (1, 0)) = \pi(1, 0) = y_0 \neq y$, by the previous step. On the other hand, if $g \notin Z$, then $gZ \neq Z$, and the closed subsets $gZ \times V$ and $Z \times V$ of X are disjoint and Z -stable under the $*$ -action, and hence separated by π . In particular, $g \cdot y_0 = \pi(g \cdot (1, 0)) = \pi(g, 0) \neq \pi(1, v) = y$ for any $g \notin Z$. Therefore we do not have $g \cdot y_0 = y$ for any $g \in G$.

Combining all the steps above, we see that the G -orbit of $y \in Y^H$ is not closed, which is a contradiction to hypothesis (i). Hence our assumption that $N_G(H)/H$ is infinite was false and we must in fact have (i) \implies (ii), as required.

For the final statement of the theorem, suppose that H is G -cr. By [32, Prop. 4.1], this means that H^0 is reductive, so we have (i) \implies (ii) by the previous arguments. Finally, suppose (ii) holds and X is any affine G -variety. Let $x \in X^H$, so that $H \subseteq G_x$, and let $g \in N_G(H)$ and $h \in H$. There exists $h_1 \in H$ such that $hg = gh_1$ and so we have $h \cdot (g \cdot x) = g \cdot (h_1 \cdot x) = g \cdot x$. Thus $g \cdot x \in X^H$ for all $g \in N_G(H)$, and hence the whole orbit $N_G(H) \cdot x \subseteq X^H$. Since $N_G(H)/H$ is finite, this orbit is a finite union of H -orbits. But each H -orbit in X^H is a singleton so $N_G(H) \cdot x$ is also finite, and therefore closed in X . Now we can apply Theorem 4.1 to deduce that the G -orbit of x is also closed, which gives (ii). \square

Remarks 4.6. (i). The original proof of (i) \implies (ii) in [19, Cor. 3] goes via a similar quotient, this time of the product of $G \times V$ by the group $N_G(H)$, where V is a non-trivial $N_G(H)/H$ -module and the action of $N_G(H)$ on $G \times V$ is similar to the action of Z in the proof above. Since the start of the proof shows that $N_G(H)^0$ is a reductive group, this quotient still exists in positive characteristic. However, the arguments in [19] are rather terse and it is not clear, at least to the author, how to guarantee in general that such a quotient has the properties we need. In particular, in the notation of the proof, we need the points y and y_0 to be distinct in Y and our argument for this relies on the specific choice of V described in the proof.

(ii). Luna's original result [19, Cor. 3] gives the equivalence (i) \iff (ii) above in characteristic 0 whenever H^0 is reductive. Example 4.7 below shows that the implication (i) \Leftarrow (ii) in Theorem 4.5 simply does not hold in general when H^0 is reductive but H is not G -cr. It has proved difficult to find any examples for which Theorem 4.5(i) holds where the subgroup H is not in fact G -cr, and it would seem reasonable to conjecture that no such examples exist (i.e., that a subgroup satisfying (i) is automatically G -cr). In any case, the equivalence (i) \iff (ii) when H is G -cr gives another example where complete reducibility is a good replacement for the reductivity hypothesis in characteristic 0.

(iii). The original proofs in [19] rest on various constructions related to Luna's celebrated Étale Slice Theorem [18]. These methods do not work in general in positive characteristic (see [1]), but in the light of the results in this paper it would be interesting to investigate whether one can extend any of the results in [1] by using G -complete reducibility, and also whether any of the many results in characteristic zero which use the Étale Slice Theorem can be approached in positive characteristic via this framework.

Example 4.7. The following example shows that the implication (i) \Leftarrow (ii) in Theorem 4.5 is false in general when H is reductive but not G -cr.

Let $p = 2$ and let $\rho : \mathrm{SL}_2(k) \rightarrow \mathrm{SL}_3(k)$ be the adjoint representation of SL_2 . Concretely, let

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

be the standard basis for $\text{Lie}(\text{SL}_2)$ and let SL_2 act by conjugation. Then, with respect to this basis, we have

$$\rho \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 & 0 & b^2 \\ ac & 1 & bd \\ c^2 & 0 & d^2 \end{pmatrix}.$$

Note that ρ does land in SL_3 since $a^2d^2 - b^2c^2 = (ad - bc)^2$ in characteristic 2. Let H be the image of ρ inside $G = \text{SL}_3$. Then H is reductive, but H is not G -cr since the representation ρ is not semisimple: the H -fixed subspace spanned by the vector h has no H -stable complement. Since H is reductive, we have $N_G(H)^0 = H^0 C_G(H)^0$, by [20, Lem. 6.8]. Direct calculation shows that $C_G(H)$ is finite and hence $N_G(H)/H$ is finite. Now the vector h is H -fixed but has a non-closed G -orbit, since if we let $\lambda \in Y(G)$ be the cocharacter defined by

$$\lambda(t) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-1} \end{pmatrix}$$

for each $t \in k^*$, then $\lambda(t) \cdot h = th$, so $\lim_{t \rightarrow 0} \lambda(t) \cdot h = 0$. It is obvious that 0 is not G -conjugate to h .

Note that this example only works in characteristic 2 because it relies on the existence of the H -fixed vector h . This is consistent with the results above, since away from characteristic 2 the image of the adjoint representation of SL_2 in SL_3 is completely reducible.

Corollary 4.8. *Suppose X is an affine G -variety and $x \in X$ is a point such that G_x contains a maximal torus of G . Then $G \cdot x$ is closed in X .*

Proof. Let T be a maximal torus of G contained in G_x ; then $x \in X^T$. But $N_G(T)/T$ is finite and any torus is G -cr, so the result follows from (ii) \implies (i) in Theorem 4.5. \square

Remark 4.9. Corollary 4.8 was proved in [16, Kap. III, 2.5, Folgerung 3] in the case where the characteristic is 0. We present it here as a corollary of Theorem 4.5, but it is also straightforward to construct a more direct proof using Kempf's Theorem 2.3.

We finish this subsection by pointing out that although the hypotheses in Theorem 4.5 seem quite restrictive, there is in fact a ready supply of subgroups satisfying them in any reductive group G . To see this, let A be any G -cr subgroup of G and set $H = N_G(A)$. It follows from [4, Thm. 3.14] that H is also G -cr. Moreover, by [4, Prop. 3.12], $N_G(H)^0 = H^0 C_G(H)^0$. Since $A \subseteq H$, we have $C_G(H) \subseteq C_G(A) \subseteq H$, and hence $N_G(H)^0 = H^0$. Therefore $N_G(H)/H$ is finite.

4.3. Some more applications to complete reducibility. A theme running through [4] and subsequent papers on complete reducibility by the same authors is the following general question: if A and H are subgroups of G with $A \subseteq H$ and H reductive, what conditions ensure that A is G -cr implies A is H -cr, and vice versa? Because of the link between complete reducibility and closed orbits in G^n explained in subsection 4.1 above, this is readily seen to be a special case of the general questions considered in this paper. Since this was one of the original motivations for the work presented here, we briefly record some of the translations of our main results into the language of complete reducibility and give a couple of other consequences in this setting.

First note that Theorem 3.11 specializes to [4, Prop. 3.19] in the setting of complete reducibility: that is, with notation as just set up, if H also contains a maximal torus of $C_G(A)$, then A is G -cr implies A is H -cr. More generally, we have:

Proposition 4.10. *Suppose H is a reductive group, and let $A \subseteq H$ be a subgroup of H .*

- (i) *If A is H -cr, then for any embedding of H as a closed subgroup of a linear algebraic group G , $HC_G(A)$ is closed in G .*
- (ii) *If there exists an embedding of H in a reductive group G such that A is G -cr, then A is H -cr if and only if $HC_G(A)$ is closed in G .*

Proof. (i). Let $\mathbf{a} \in H^n$ be a generic tuple for A . Then A is H -cr implies that $H \cdot \mathbf{a}$ is closed in H^n . Since, for any embedding $H \subseteq G$, H^n is closed in G^n , we have that $H \cdot \mathbf{a}$ is closed in $G \cdot \mathbf{a}$ for all such embeddings. Therefore, by Theorem 3.3(i), $HG_{\mathbf{a}} = HC_G(A)$ is closed in G .

(ii). Using a generic tuple for A again, this becomes a direct application of Theorem 3.3(ii). \square

The notions of reductive pairs from [27, §3] and separability from [4, Def. 3.27] have proved useful in the study of complete reducibility, see [4, §3.5], [6], [3] for example. Recall that a pair (G, H) of reductive

groups with $H \subseteq G$ is called a *reductive pair* if $\mathrm{Lie}(H)$ splits off as a direct H -module summand of $\mathrm{Lie}(G)$, where H acts via the adjoint action of G on $\mathrm{Lie}(G)$, and a subgroup $A \subseteq G$ is called *separable in G* if

$$\mathrm{Lie}(C_G(A)) = c_{\mathrm{Lie}(G)}(A) := \{X \in \mathrm{Lie}(G) \mid \mathrm{Ad}_G(a)(X) = X \text{ for all } a \in A\}.$$

With these definitions in hand, we have the following result:

Proposition 4.11. *Suppose (G, H) is a reductive pair. Let A be a separable subgroup of G contained in H . Then $HC_G(A)$ is closed in G .*

Proof. Let $\mathbf{a} \in H^n$ be a generic tuple for A . Then $C_G(A) = G_{\mathbf{a}}$, and since A is separable in G the orbit map $G \rightarrow G \cdot \mathbf{a}$ is separable. Now Richardson’s “tangent space argument” [27, §3] (generalized to n -tuples in [33]) shows that $G \cdot \mathbf{a} \cap H^n$ decomposes into finitely many H -orbits, each of which is closed in $G \cdot \mathbf{a} \cap H^n$. Since one of these orbits is $H \cdot \mathbf{a}$, we can conclude that $H \cdot \mathbf{a}$ is closed in $G \cdot \mathbf{a} \cap H^n$, and hence in $G \cdot \mathbf{a}$. Therefore, $HG_{\mathbf{a}} = HC_G(A)$ is closed in G by Theorem 3.3. \square

Remark 4.12. Note that every pair (G, H) of reductive groups with $H \subseteq G$ is a reductive pair in characteristic 0 and the separability hypothesis is also automatic. In characteristic $p > 0$, every subgroup of G is separable as long as p is “very good” for G , see [6, Thm. 1.2].

As a final remark, we note that the results presented in this section can all be translated into statements about n -tuples and orbits in G^n and there are also analogous results which hold for Lie subalgebras of $\mathrm{Lie}(H)$ and $\mathrm{Lie}(G)$, etc. For details of how to make such translations, see [8, §5] for example.

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